

CS51

AD648394

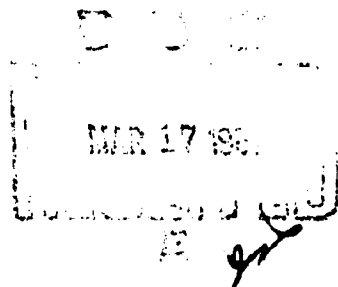


NUMERICAL CALCULATION OF TRANSONIC FLOW PATTERNS

BY

S. BERGMAN, J. G. HERRIOT and T. G. KURTZ

**TECHNICAL REPORT NO. CS51
OCTOBER 14, 1966**



LIBRARY COPY

JAN 3 1967

**LEWIS LIBRARY, NASA
CLEVELAND, OHIO**

**COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY**



NUMERICAL CALCULATION OF TRANSONIC FLOW PATTERNS

by

Stefan Bergman, John G. Herriot and Thomas G. Kurtz

1. Introduction.

In order to obtain flow patterns of incompressible fluids, one often uses the well-known fact that the real and imaginary parts of an analytic function $f(z)$, $z = x + iy$, are solutions of the Laplace equation. In particular $\text{Im } f(z)$ is the stream function $\psi(x,y)$ of a possible flow pattern. If one wishes to investigate methods for determining the analytic function $f(z)$ which gives a flow in a prescribed channel or around a given profile, one is led to a boundary value problem which as a rule has a solution.

As a first step in the calculation of desired flow patterns it is instructive to compute the flow patterns which correspond to various known functions $f(z)$. These flow patterns may provide insight which suggests how to form combinations of stream functions which yield a desired flow. For example R. Legendre [11, 12] showed that various transcendental functions generate interesting flow patterns and he used these in the solution of certain flow problems.

In some methods for the numerical solution of boundary value problems, one first obtains a set of particular solutions, ψ_v , of the differential equation and then one determines a set of coefficients A_v so that

Prepared under NSP Grant GP 4399 and Air Force Grant AF-OSR-1047-66
at Stanford University.

$$(1.1) \quad \psi^{(N)} = \sum_{v=1}^N A_v \psi_v$$

approximates in some "best" sense the prescribed boundary values.

The aim of the present paper is to describe procedures for computing families of transonic flow patterns in a compressible fluid. The partial differential equation for the stream function of such flows is nonlinear when considered in the physical plane but is linear if the equation is considered in the so-called hodograph plane (or in the H, θ plane which is related to the hodograph plane as described below). For this reason we shall determine a number of solutions of the differential equation for the stream function in the H, θ plane and we shall also calculate the corresponding flow patterns in the physical plane. It appears to be even more important to determine families of transonic flow patterns than it was to determine incompressible fluid flows because it can be shown that a flow pattern with boundary C does not necessarily exist for a given contour C . Moreover, while the determination of solutions of the Laplace equation is immediate, the determination of solutions of the compressibility equation in the H, θ plane (equation (1.6) or (1.7)) requires the use of methods which have been developed during the last twenty-five years. The numerical evaluation of these solutions poses problems which are only now being investigated.

We proceed now to a more detailed description of the method to be used in the present paper. As noted above, it is well-known that the partial differential equation satisfied by the stream function ψ of a two-dimensional compressible fluid flow when considered in the physical

plane (i.e. when ψ is considered as a function of the coordinates x, y of the plane in which the flow takes place) is non-linear. If, however, we consider the equation in the hodograph plane, i.e. if we consider ψ as a function of the speed v and the angle θ which the velocity vector makes with a fixed direction (say with the x -axis), then as Chaplygin [9] and Molenbroek [14] have shown, ψ satisfies the linear partial differential equation

$$(1.2) \quad \frac{1-M^2}{2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{v}{\rho} \frac{\partial}{\partial v} \left(\frac{v}{\rho} \frac{\partial \psi}{\partial v} \right) = 0$$

Here ρ denotes the density which is related to the pressure p by the relation $p = c\rho^k$ where c and k are constants ($k = 1.4$ for air). The Mach number M and the density ρ are related to the velocity v according to the relations

$$(1.3) \quad \rho = \left(1 - \frac{k-1}{2} v^2 \right)^{1/(k-1)}$$

$$(1.4) \quad M = v / \left(1 - \frac{k-1}{2} v^2 \right)^{1/2}$$

The denominator in (1.4) being the local velocity of sound, M will be smaller or larger than one according as the flow is subsonic or supersonic. Instead of the velocity v we may introduce the variable

$$(1.5) \quad H = \int_{v_1}^v \frac{\rho}{v} dv = \int_{v_1}^v \left(1 - \frac{k-1}{2} v^2 \right)^{1/(k-1)} \frac{dv}{v}$$

where v_1 denotes the velocity corresponding to $M = 1$. If one introduces the potential Φ as well as the stream function Ψ , then the differential equations connecting these quantities assume the form

$$(1.6) \quad \frac{\partial \Phi}{\partial \theta} = \frac{\partial \Psi}{\partial H}, \quad \frac{\partial \Phi}{\partial H} = -\mathcal{L}(H) \frac{\partial \Psi}{\partial \theta}, \quad \mathcal{L}(H) = \frac{1-M^2}{2}$$

(See [1, 3, 4, 6, 13].)

From (1.6) we deduce at once that the differential equation for Ψ may be written in the form

$$(1.7) \quad \frac{\partial^2 \Psi}{\partial H^2} + \mathcal{L}(H) \frac{\partial^2 \Psi}{\partial \theta^2} = 0$$

Equation (1.7) is a differential equation of mixed type since $\mathcal{L}(H) > 0$ for $H < 0$ (i.e. for $v < v_1$, $M < 1$) while $\mathcal{L}(H) < 0$ for $H > 0$ (i.e. for $v > v_1$, $M > 1$).

A representation of the solutions of equations (1.6) was obtained by Bergman [1, 2]. In a joint investigation Bers and Gelbart [7, 8] obtained independently of Bergman the same solutions. In their work the existence of these solutions follows as a consequence of their approach to more general partial differential equations by complex variable methods. Bers and Gelbart call the complex functions which they introduced by the name of Sigma-Monogenic functions. This method for obtaining a set of particular solutions of (1.6) and (1.7) is a generalization of the method of obtaining harmonic polynomials by taking real and imaginary parts of the analytic functions $(\theta + iH)^n$. These particular solutions $\Phi_{n,j}(H, \theta)$ and $\Psi_{n,j}(H, \theta)$, $j = 1, 2$, $n = 0, 1, \dots$ which satisfy (1.6) (and hence

(1.7) in the case of $\psi_{n,j}(H, \theta)$ may be written in the form

$$\begin{aligned}
 \varphi_{n,1}(H, \theta) + i\psi_{n,1}(H, \theta) &= (\theta + iH)^{[n]} \\
 (1.8) \quad &\equiv [\theta^n - 2! \binom{n}{2} \theta^{n-2} p_2(H) + 4! \binom{n}{4} \theta^{n-4} p_4(H) + \dots] \\
 &\quad + i[1! \binom{n}{1} \theta^{n-1} s_1(H) - 3! \binom{n}{3} \theta^{n-3} s_3(H) + \dots]
 \end{aligned}$$

$$\begin{aligned}
 \varphi_{n,2}(H, \theta) + i\psi_{n,2}(H, \theta) &= i \odot (\theta + iH)^{[n]} \\
 (1.9) \quad &\equiv -[1! \binom{n}{1} \theta^{n-1} p_1(H) - 3! \binom{n}{3} \theta^{n-3} p_3(H) + \dots] \\
 &\quad + i[\theta^n - 2! \binom{n}{2} \theta^{n-2} s_2(H) + 4! \binom{n}{4} \theta^{n-4} s_4(H) - \dots]
 \end{aligned}$$

where

$$\begin{aligned}
 p_0(H) &= 1, \quad p_1(H) = \int_{H_0}^H \ell(H_1) dH_1 \\
 (1.10) \quad p_m(H) &= \int_{H_0}^H \ell(H_m) \int_{H_0}^{H_m} \int_{H_0}^{H_m-1} \ell(H_{m-2}) \int_{H_0}^{H_{m-2}} \dots dH_1 dH_2 \dots dH_m \\
 &\quad m = 2, 3, \dots
 \end{aligned}$$

$$\begin{aligned}
 s_0(H) &= 1, \quad s_1(H) = H - H_0 = \int_{H_0}^H dH_1 \\
 (1.11) \quad s_m(H) &= \int_{H_0}^H \int_{H_0}^{H_m} \ell(H_{m-1}) \int_{H_0}^{H_{m-1}} \int_{H_0}^{H_{m-2}} \ell(H_{m-2}) \dots dH_1 dH_2 \dots dH_m \\
 &\quad m = 2, 3, \dots
 \end{aligned}$$

One easily deduces that $p_m(H)$ and $s_m(H)$ satisfy the recurrence relations

$$(1.10a) \quad p_m(H) = \int_{H_0}^H \mathcal{L}(H_m) \int_{H_0}^{H_m} p_{m-2}(H_{m-1}) dH_{m-1} dH_m$$

and

$$(1.11a) \quad s_m(H) = \int_{H_0}^H \int_{H_0}^{H_m} \mathcal{L}(H_{m-1}) s_{m-2}(H_{m-1}) dH_{m-1} dH_m$$

respectively. It will be convenient to choose $H_0 = 0$ corresponding to $M = 1$.

In Section 2 we approximate the function $\mathcal{L}(H)$ by polynomials and calculate approximations $\tilde{\psi}_{n,j}$ to the stream functions $\psi_{n,j}$. In Section 3 we then form approximate stream functions which are linear combinations of these $\tilde{\psi}_{n,j}$ and we determine the flow patterns in the physical plane corresponding to these approximate stream functions.

The method of integral operators [5, Chap. 5] provides another method for generating particular solutions of the compressible flow equation. Some numerical calculations using this method have been carried out by Stark [18, 19].

Essentially the same partial differential equations arise in the study of magnetohydrodynamics. For the case of potential flow in a magnetic field perpendicular to the plane of the flow, and under some additional hypotheses, Nazarov [15, 16] has shown that the differential equations are essentially the same as the equation (1.2) for mixed compressible flow. He then showed how to determine various flow patterns using the method of integral operators.

2. Calculation of Stream Functions

In order to calculate the stream functions $\psi_{n,1}$ and $\psi_{n,2}$ given by (1.8) and (1.9) it is necessary to evaluate the functions $s_m(H)$ which are given by (1.11) in terms of multiple integrals involving the function $l(H)$. Because of the nature of the function $l(H)$ these integrals cannot be evaluated exactly. One possible method for the evaluation of these integrals is to approximate $l(H)$ in suitable intervals by polynomials in H . The intervals chosen were

$$J_1: -1.0 \leq H \leq 0 \quad \text{corresponding to} \quad .284 \leq M \leq 1$$

(2.1)

$$J_2: 0 \leq H \leq 0.2 \quad \text{corresponding to} \quad 1 \leq M \leq 1.78$$

The polynomial used to approximate $l(H)$ in J_κ is denoted by

$$(2.2) \quad \chi_\kappa(H) = \sum_{v=0}^{10} A_v^{(\kappa)} \eta_\kappa^v, \quad \eta_1 = H, \quad \eta_2 = 5H, \quad \kappa = 1, 2.$$

In order to determine the coefficients of the approximating polynomial in these intervals we need an algorithm for the calculation of $l(H)$ as a function of H . We consider the special case of air for which $k = 1.4$. Since ρ and M are functions of v as shown by equations (1.3) and (1.4), then $l(H)$ can be expressed as a function of v using its definition in (1.6). In this special case the integral in (1.5) can be evaluated in closed form and yields

$$\begin{aligned}
 H = & \left\{ (1-0.2v^2)^{1/2} \left[\frac{(1-0.2v^2)^2}{5} + \frac{1-0.2v^2}{3} + 1 \right] \right. \\
 (2.3) \quad & \left. - \frac{1}{2} \ln \left(\frac{1+(1-0.2v^2)^{1/2}}{1-(1-0.2v^2)^{1/2}} \right) \right\} \Big|_{v_1}^v
 \end{aligned}$$

where $v_1 = \sqrt{5/6}$. From (2.3) we can devise an algorithm for the calculation of v corresponding to any value of H in J_1 or J_2 and then $l(H)$ can be calculated at once.

We then make use of the Remez algorithm [17] as adapted for the Burroughs B5500 computer by Golub and Smith [10] to determine the best tenth-degree polynomials which approximate $l(H)$ in the Chebyshev sense separately on the intervals J_1 and J_2 . The values of the coefficients $A_v^{(\kappa)}$, $\kappa = 1, 2$, $v = 0, 1, \dots, 10$ are given in Table 1. Values of H , v , M , $\tilde{l}_\kappa(H)$, $l(H)$ and $\tilde{l}_\kappa(H) - l(H)$ are given in Table 2 for a representative set of values of H . From the approximation algorithm we know that

$$\begin{aligned}
 \max_{J_1} |l(H) - \tilde{l}_1(H)| &\leq 1.7 \times 10^{-4} \\
 (2.4) \quad \max_{J_2} |l(H) - \tilde{l}_2(H)| &\leq 2.6 \times 10^{-3}
 \end{aligned}$$

The values given in Table 2 confirm these bounds.

The function $l(H)$ becomes very large and negative when H exceeds 0.2. Indeed as $H \rightarrow 0.2513$, $M \rightarrow \infty$, $v \rightarrow \sqrt{5}$ and $l(H) \rightarrow -\infty$. It was therefore practically impossible to obtain a good polynomial approximation for $H > 0.2$.

Table 1. Values of the coefficients $A_v^{(K)}$ of
the approximations to $L(H)$.

$\begin{smallmatrix} K \\ v \end{smallmatrix}$	1	2
0	0.0001711222	-0.0025985374
1	-9.3681945493	-1.3189899758
2	-55.284916968	-22.598654078
3	-248.40627057	280.04321217
4	-840.40978809	-2002.2845890
5	-2052.0104328	8142.5278038
6	-3488.0712341	-20236.235647
7	-3982.8542755	31196.920595
8	-2896.2958023	-29181.568535
9	-1207.8174204	15188.597028
10	-219.39926509	-3388.9118171

TABLE 2. VALUES OF $I(H)$ AND ITS APPROXIMATING POLYNOMIALS

SUBSONIC					
H	γ	M	$I(H)$	$\tilde{I}_1(H)$	$I(H) - \tilde{I}_1(H)$
-1.0000	0.281675	0.283937	0.995929	0.995758	0.000171
-0.9500	0.296755	0.299404	0.994943	0.994945	-0.000002
-0.9000	0.312719	0.315823	0.993706	0.993550	0.000155
-0.8500	0.329633	0.333274	0.992148	0.992223	-0.000074
-0.8000	0.347570	0.351846	0.990181	0.990346	-0.000165
-0.7500	0.366611	0.371640	0.987685	0.987693	-0.000008
-0.7000	0.386848	0.392770	0.984504	0.984349	0.000155
-0.6500	0.408386	0.415372	0.980431	0.980299	0.000132
-0.6000	0.431344	0.439600	0.975181	0.975218	-0.000037
-0.5500	0.455859	0.465638	0.968369	0.968534	-0.000165
-0.5000	0.482091	0.493701	0.959460	0.959579	-0.000119
-0.4500	0.510227	0.524053	0.947697	0.947646	0.000051
-0.4000	0.540492	0.557009	0.931997	0.931828	0.000169
-0.3500	0.573154	0.592964	0.910772	0.910672	0.000100
-0.3000	0.608543	0.632414	0.881631	0.881721	-0.000089
-0.2500	0.647071	0.675994	0.840877	0.841045	-0.000167
-0.2000	0.689261	0.724542	0.782574	0.782577	-0.000003
-0.1500	0.735800	0.779194	0.696775	0.696605	0.000170
-0.1000	0.787612	0.841544	0.565895	0.565909	-0.000015
-0.0500	0.845996	0.913932	0.356635	0.356757	-0.000122
0.0000	0.912871	1.000000	0.000000	0.000171	-0.000171

SUPERSONIC					
H	γ	M	$I(H)$	$\tilde{I}_2(H)$	$I(H) - \tilde{I}_2(H)$
0.0000	0.912871	1.000000	0.000000	-0.002599	0.002599
0.0200	0.942622	1.039500	-0.214281	-0.216638	0.002357
0.0400	0.974465	1.082684	-0.493622	-0.491117	-0.002505
0.0600	1.008744	1.130295	-0.865912	-0.868276	0.002365
0.0800	1.045902	1.183327	-1.375578	-1.374987	-0.000591
0.1000	1.086529	1.243155	-2.096940	-2.095146	-0.001794
0.1200	1.131437	1.311754	-3.162109	-3.164673	0.002565
0.1400	1.181796	1.392111	-4.825058	-4.823672	-0.001385
0.1600	1.239389	1.489049	-7.627815	-7.627813	-0.000002
0.1800	1.307166	1.611131	-12.911118	-12.910861	-0.000257
0.2000	1.390636	1.775837	-24.834790	-24.832192	-0.002598

We may now calculate exactly the solutions $\tilde{\psi}_{n,1}$ and $\tilde{\psi}_{n,2}$ of the differential equation

$$(2.5) \quad \tilde{\psi}_{HH} + \chi(H)\tilde{\psi}_{\theta\theta} = 0, \quad \chi(H) = \chi_K(H) \text{ in } J_K$$

which approximates the differential equation (1.7). If we use (1.11) and denote by $\tilde{s}_m(H)$ the polynomials which approximate $s_m(H)$ we find that

$$(2.6) \quad \begin{aligned} \tilde{s}_0(H) &= 1, & \tilde{s}_1(H) &= H \\ \tilde{s}_2(H) &= \int_0^H \int_0^{H_2} \chi(H_1) dH_1 dH_2 \\ \tilde{s}_3(H) &= \int_0^H \int_0^{H_3} \chi(H_2) \int_0^{H_2} dH_1 dH_2 dH_3 \\ &\dots\dots\dots \\ \tilde{s}_m(H) &= \int_0^H \int_0^{H_m} \chi(H_{m-1}) \tilde{s}_{m-2}(H_{m-1}) dH_{m-1} dH_m \end{aligned}$$

If we make use of equation (2.2) and denote $\tilde{s}_m(H)$ by $\tilde{s}_m^{(\kappa)}(H)$ in J_K we obtain

$$(2.7) \quad \tilde{s}_2^{(\kappa)}(H) = H^2 \sum_{v=0}^{10} c_v^{(2,\kappa)} \eta_K^v, \quad c_v^{(2,\kappa)} = \frac{A_v^{(\kappa)}}{(v+1)(v+2)}, \quad \kappa = 1, 2$$

$$(2.8) \quad \tilde{s}_3^{(\kappa)}(H) = H^3 \sum_{v=0}^{10} c_v^{(3,\kappa)} \eta_K^v, \quad c_v^{(3,\kappa)} = \frac{A_v^{(\kappa)}}{(v+2)(v+3)}, \quad \kappa = 1, 2$$

The general expression for $\tilde{s}_m(H)$ takes the form

$$(2.9) \quad \tilde{s}_m^{(\kappa)}(H) = H^m \sum_{v=0}^{10[\frac{m}{2}]} C_v^{(m,\kappa)} \eta_K^v \quad \kappa = 1, 2$$

where

$$(2.10) \quad C_v^{(m,\kappa)} = \sum_{\{ \ell | 0 \leq \ell \leq 10, 0 \leq v-\ell \leq 10[\frac{m-2}{2}] \}} \frac{A_\ell^{(\kappa)} C_{v-\ell}^{(m-2,\kappa)}}{(v+m-1)(v+m)}$$

Note that $[\frac{m}{2}] = \frac{m}{2}$ if m is even but $[\frac{m}{2}] = \frac{m-1}{2}$ if m is odd.

The approximate stream functions $\tilde{\psi}_{n,1}$ and $\tilde{\psi}_{n,2}$ can now be calculated using equations (1.8) and (1.9) with $s_m(H)$ replaced by $\tilde{s}_m(H)$.

It is seen from Table 1 that the coefficients of the polynomials $\chi_K(H)$ are very large; they are of constant sign in $\chi_1(H)$ which is used for $H < 0$ and of alternating sign in $\chi_2(H)$ which is used for $H > 0$. Hence in both cases the evaluation of these functions and of the functions $\tilde{s}_m^{(\kappa)}(H)$ will involve large terms of alternating sign and this may result in loss of several significant digits. The magnitude of the error given in (2.4) indicates that single precision arithmetic (about 11 decimal digits) on the Stanford Burroughs B5500 computer is adequate for the calculation of $\chi_K(H)$. However, for the evaluation of $\tilde{s}_m^{(\kappa)}(H)$ and the subsequent calculation of $\tilde{\psi}_{n,j}$, single precision arithmetic was found to be inadequate. Hence these calculations were carried out in double precision arithmetic (78 bit arithmetic equivalent to about 22 decimal digits) on the B5500. Some typical values of $\tilde{\psi}_{n,j}$ are given in Table 3. (Since $\tilde{\psi}_{1,1} = H$, $\tilde{\psi}_{2,1} = 2\theta H$, $\tilde{\psi}_{1,2} = \theta$, these are not included in Table 3.)

It is important to investigate how well the approximate solutions $\tilde{\Psi}_{n,j}$ satisfy the differential equation (1.6). We therefore study the behavior of $S(\tilde{\Psi}_{n,j})$ where

$$(2.11) \quad S(\tilde{\Psi}) = \frac{\partial^2 \tilde{\Psi}}{\partial H^2} + L(H) \frac{\partial^2 \tilde{\Psi}}{\partial \theta^2}$$

If we choose $\tilde{\Psi}$ to be a solution of (2.5) then it follows at once that

$$(2.12) \quad S(\tilde{\Psi}) = [L(H) - \chi(H)] \frac{\partial^2 \tilde{\Psi}}{\partial \theta^2}.$$

By formally differentiating equations (1.8) and (1.9) we can find formulas for $\partial^2 \tilde{\Psi}_{n,j} / \partial \theta^2$ and these can be used to evaluate $S(\tilde{\Psi}_{n,j})$. These calculations were carried out in double precision arithmetic on the B5500. Typical values are given in Table 3.

It is seen that the values of $S(\tilde{\Psi}_{n,j})$ in Table 3 are small. It would be more meaningful to compare them with the magnitude of $\tilde{\Psi}_{n,j}$ and still better to consider the average values over some region G of the H, θ plane. We therefore introduce the quantity

$$(2.13) \quad M_G(\tilde{\Psi}) = \left[\frac{\int_G \int |S(\tilde{\Psi})|^2 dH d\theta}{\int_G \int |\tilde{\Psi}|^2 dH d\theta} \right]^{1/2}$$

If one chooses

$$G = \{H, \theta \mid -1.0 \leq H \leq .2, \quad 0 \leq \theta \leq 2\}$$

an extensive calculation shows that

$$M_G(\tilde{\gamma}_{n,1}) \leq 0.01, \quad n \leq 7$$

$$M_G(\tilde{\gamma}_{n,2}) \leq 0.01, \quad n \leq 5$$

TABLE 3. - VALUES OF $\tilde{v}_{n,j}$ AND $S(\tilde{v}_{n,j})$.

PSI(3, 1)

THETA		0.50	1.00	2.00
H	$\tilde{v}$	$S(\tilde{v})$	$\tilde{v}$	$S(\tilde{v})$
-1.00	0.1633700	-0.0010265	-2.0866300	-0.0010265
-0.90	-0.0196152	-0.0008397	-2.0446152	-0.0008397
-0.80	-0.1469456	0.0007931	-1.9259456	0.0007931
-0.70	-0.2307471	-0.0006517	-1.8057471	-0.0006517
-0.60	-0.2712086	0.0001343	-1.6212086	0.0001343
-0.50	-0.2765697	0.0003584	-1.4015697	0.0003584
-0.40	-0.2531522	-0.0004052	-1.1531522	-0.0004052
-0.30	-0.2073766	0.0001605	-0.8823766	0.0001605
-0.20	-0.1457304	0.0000032	-0.5957304	0.0000032
-0.10	-0.0746577	0.0000084	-0.2996577	0.0000084
0.00	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0150008	0.0002828	0.0600008	0.0002828
0.06	0.0450781	0.0008513	0.1800781	0.0008513
0.10	0.0757382	-0.0010744	0.3007382	-0.0010744
0.14	0.1066399	-0.0011564	0.4236399	-0.0011564
0.18	0.1468513	-0.0002686	0.5538513	-0.0002686

PSI(4, 1)

THETA		0.50	1.00	2.00
H	$\tilde{v}$	$S(\tilde{v})$	$\tilde{v}$	$S(\tilde{v})$
-1.00	1.3267400	-0.0020530	-0.3465201	-0.0041061
-0.90	0.8607497	-0.0016795	-0.9784606	-0.0033590
-0.80	0.5021087	0.0015863	-1.3957825	0.0031726
-0.70	0.2385057	-0.0013034	-1.6229855	-0.0026067
-0.60	0.0575827	0.0002667	-1.6848346	0.0005373
-0.50	-0.0531394	0.0007169	-1.6062788	0.0014338
-0.40	-0.1063044	-0.0008105	-1.4126089	-0.0016210
-0.30	-0.1147532	0.0003211	-1.1295064	0.0006422
-0.20	-0.0914608	0.0000063	-0.7829216	0.0000126
-0.10	-0.0493153	0.0000168	-0.3988307	0.0000335
0.00	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0109016	0.0005656	0.0800033	0.0011312
0.06	0.0301561	0.0017026	0.2403123	0.0034052
0.10	0.0514764	-0.0021488	0.4029527	-0.0042976
0.14	0.0772797	-0.0023129	0.5745595	-0.0046257
0.18	0.1177024	-0.0005372	0.7754052	-0.0010745

PSI(5, 1)

THETA		0.50	1.00	2.00
H	$\tilde{v}$	$S(\tilde{v})$	$\tilde{v}$	$S(\tilde{v})$
-1.00	1.1593300	0.0005591	1.3221048	-0.0071378
-0.90	0.8937381	-0.0000610	1.5903744	-0.0063571
-0.80	0.6311412	0.0004922	0.2640490	0.0064408
-0.70	0.3972384	-0.0007161	-0.6771151	-0.0056037
-0.60	0.2080588	0.0002024	-1.2435060	0.0012099
-0.50	0.0711690	0.0006609	-1.5343538	0.0033492
-0.40	-0.0131503	-0.0008549	-1.5367919	-0.0038943
-0.30	-0.0506833	0.0003609	-1.3247579	0.0015740
-0.20	-0.0519131	0.0000077	-0.9573910	0.0000314
-0.10	-0.0303953	0.0000209	-0.4965778	0.0000838
0.00	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0042520	0.0007070	0.1000081	0.0028280
0.06	0.0189453	0.0021320	0.3007808	0.0085168
0.10	0.0331006	-0.0027124	0.5073870	-0.0107705
0.14	0.0529404	-0.0028913	0.7364894	-0.0114446
0.18	0.0719094	-0.0007404	1.0395441	-0.0027550

TABLE 3. - VALUES OF $\tilde{v}_{n,j}$ AND $S(\tilde{v}_{n,j})$

PSI(6. 1)

THETA		0.50		1.00		2.00	
H	$\tilde{v}$	$S(\tilde{v})$	$\tilde{v}$	$S(\tilde{v})$	$\tilde{v}$	$S(\tilde{v})$	
-1.00	-0.3388599	0.0069099	7.3978298	-0.0017780	-55.5999433	-0.1267378	
-0.90	0.0792901	0.0040137	4.9268528	-0.0045647	-73.5001133	-0.1098984	
-0.80	0.2381519	-0.0024890	2.7421193	0.0069191	-84.3892369	0.1090151	
-0.70	0.3454309	0.0011102	0.9671948	-0.0075549	-88.7552657	-0.0933119	
-0.60	0.1802196	-0.0000645	-0.3326905	0.0018861	-87.2104182	0.0198924	
-0.50	0.8963554	0.0001905	-1.1433347	0.0057576	-80.4750336	0.0545288	
-0.40	0.0263103	-0.0005385	-1.4946627	-0.0071557	-49.3675915	-0.0629409	
-0.30	-0.0151670	0.0003071	-1.4534831	0.0030223	-54.7921591	0.0253101	
-0.20	-0.0270873	0.0000072	-1.1151304	0.0000618	-37.7179078	0.0005026	
-0.10	-0.0178975	0.0000207	-0.5931600	0.0001672	-19.1452399	0.0013407	
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	
0.02	0.0037520	0.0007071	0.1200183	0.0056562	3.8401302	0.0452483	
0.04	0.0114455	0.0021393	0.3615621	0.0170483	11.5324931	0.1362535	
0.06	0.0206111	-0.0027653	0.6147948	-0.0216467	19.3181711	-0.1722218	
0.08	0.0356219	-0.0031917	0.9133418	-0.0237300	27.4636675	-0.1862320	
0.10	0.0714722	-0.0008782	1.3632139	-0.0057857	36.7855837	-0.0438049	

PSI(7. 1)

THETA		0.50		1.00		2.00	
H	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$	
-1.00	-1.6603979	0.0083325	8.6363088	0.0238747	-3.9749361	-0.3181466	
-0.90	-0.7746986	0.0058374	7.2288383	0.0103875	-74.7929822	-0.3020606	
-0.80	-0.2621240	-0.0043801	5.1433063	-0.0018325	-126.3823142	0.3206572	
-0.70	-0.0130929	0.0025887	2.9342525	-0.0044127	-158.8205474	-0.2890194	
-0.60	0.0700845	-0.0003261	0.9924421	0.0019802	-172.9815309	0.0640997	
-0.50	0.0661587	-0.0003571	-0.4432540	0.0074998	-170.4427244	0.1810654	
-0.40	0.0316460	-0.0000933	-1.2704092	-0.0108986	-153.4073160	-0.2136858	
-0.30	0.0005884	0.0001899	-1.5039546	0.0049626	-124.6141584	0.0872686	
-0.20	-0.0129907	0.0000057	-1.2523910	0.0001058	-87.2143335	0.0017496	
-0.10	-0.0101945	0.0000178	-0.6880418	0.0002915	-44.6083866	0.0046885	
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	
0.02	0.0021893	0.0006188	0.1400285	0.0098988	8.9604557	0.1583705	
0.04	0.0067338	0.0018816	0.4227346	0.0298733	26.9237298	0.4770422	
0.06	0.0125796	-0.0024894	0.7259455	-0.0381597	45.2138172	-0.6038872	
0.08	0.0237533	-0.0030611	1.1093031	-0.0425850	64.7659495	-0.6560264	
0.10	0.0554773	-0.0009602	1.7665290	-0.0108592	88.4834344	-0.1562217	

PSI(8. 1)

THETA		0.50		1.00		2.00	
H	Ψ	$S(\Psi)$	Ψ	$S(\Psi)$	Ψ	$S(\Psi)$	
-1.00	-1.2978519	-0.0032399	3.3942315	0.0708942	780.5541163	-0.5326284	
-0.90	-0.8603520	0.0006907	6.1335915	0.0429059	50.3864492	-0.6400668	
-0.80	-0.4549513	-0.0022037	6.1176754	-0.0253736	-119.2155195	0.7808800	
-0.70	-0.1746287	0.0021328	4.5614798	0.0084041	-242.3325237	-0.7712069	
-0.60	-0.0243249	-0.0003766	2.4492688	0.0004952	-316.8089715	0.1822394	
-0.50	0.0243178	0.0006447	0.4951412	0.0076500	-343.9154953	0.5384575	
-0.40	0.0224257	0.0002468	-0.8669683	-0.0141331	-328.0002885	-0.6559222	
-0.30	0.0053674	0.0000758	-1.4681934	0.0072598	-276.0623392	0.2736735	
-0.20	-0.0056309	0.0000040	-1.3657653	0.0001643	-197.1878526	0.0055587	
-0.10	-0.0056587	0.0000140	-0.7808918	0.0004643	-101.7870372	0.0149860	
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	
0.02	0.0012514	0.0004952	0.1600456	0.0158389	20.4814581	0.5067926	
0.04	0.0038873	0.0015157	0.4843776	0.0478801	61.5799526	1.5271969	
0.06	0.0075784	-0.0020668	0.8416296	-0.0616502	103.7251477	-1.9371814	
0.08	0.0157438	-0.0027463	1.3289317	-0.0704146	149.9233274	-2.1173111	
0.10	0.0430116	-0.0009957	2.2740323	-0.0189875	209.6047726	-0.5123954	

TABLE 3. - VALUES OF $\tilde{\psi}_{n,j}$ AND $S(\tilde{\psi}_{n,j})$.

PSI(0, 1)

THETA		0.50		1.00		2.00
H	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	0.5594215	-0.0204113	-9.5307435	0.1064768	1071.9889079	-0.0485403
-0.90	-0.1351562	-0.0086734	-0.0295244	0.0809407	561.7200719	-0.8374023
-0.80	-0.2530828	0.0031185	4.3397891	-0.0611904	103.7840640	1.5035659
-0.70	-0.1683341	-0.0001463	5.0482636	0.0327807	-265.2167729	-1.7743036
-0.60	-0.0644360	-0.0001883	3.6595486	-0.0026664	-522.7875642	0.4647486
-0.50	-0.0046504	-0.0005691	1.5419578	0.0038132	-459.9505955	1.4662673
-0.40	0.0111051	0.0003847	-0.3078839	-0.0154449	-680.2862579	-1.8650318
-0.30	0.0052635	-0.0000038	-1.3426341	0.0096581	-598.4482406	0.8002510
-0.20	-0.0020976	0.0000025	-1.4522642	0.0002373	-438.0219187	0.0165259
-0.10	-0.0030751	0.0000103	-0.8713846	0.0006924	-228.5618619	0.0448937
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0007042	0.0003715	0.1800684	0.0237599	46.0843744	1.5204035
0.04	0.0022126	0.0011465	0.5465705	0.0719754	138.6599230	4.5840730
0.10	0.0045258	-0.0016219	0.9626437	-0.0935950	234.3789421	-5.8293474
0.14	0.0104338	-0.0023545	1.5772621	-0.1099573	342.3111736	-8.197850
0.18	0.0333159	-0.0009941	2.9161851	-0.0316366	491.2786381	-1.5845289

PSI(10, 1)

THETA		0.50		1.00		2.00
H	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	2.0252775	-0.0198234	-25.6910180	0.0527421	3035.8488573	4.0146562
-0.90	0.6780410	-0.0120396	-10.5880793	0.0858524	2027.1717658	0.7052990
-0.80	0.1032402	0.0067658	-0.7645960	-0.0909781	970.5230476	1.7729059
-0.70	-0.0563278	-0.0024386	3.7203833	0.0636997	16.5405824	-3.3840980
-0.60	-0.0511046	0.0000884	4.2031476	-0.0082255	-726.5200493	1.0639614
-0.50	-0.0154502	-0.0002615	2.5210694	-0.0053244	-1195.2328436	3.6982514
-0.40	0.0030272	0.0003408	0.3630294	-0.0131751	-1370.3551355	-4.9845324
-0.30	0.0036591	-0.0000431	-1.1285843	0.0117856	-1273.2501647	2.2160320
-0.20	-0.0005634	0.0000013	-1.5093863	0.0003235	-959.1194803	0.0467044
-0.10	-0.0016409	0.0000071	-0.9592010	0.0009824	-506.7507491	0.1280476
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0003914	0.0002654	0.2000976	0.0339455	102.4124982	4.3440983
0.04	0.0012458	0.0008273	0.4093934	0.1030885	308.4000033	13.1058642
0.10	0.0026467	-0.0012213	1.0898966	-0.1356375	523.3804104	-16.7164031
0.14	0.0068949	-0.0019532	1.8599177	-0.1646594	773.4422103	-18.5760336
0.18	0.0257850	-0.0009640	3.7311824	-0.0509112	1142.7453078	-4.6919436

PSI(2, 2)

THETA		0.50		1.00		2.00
H	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	-0.5216650	0.0003422	0.2283350	0.0003422	3.2283350	0.0003422
-0.90	-0.3586351	0.0003110	0.3911649	0.0003110	3.3911649	0.0003110
-0.80	-0.2158743	-0.0003305	0.5341237	-0.0003305	3.5341237	-0.0003305
-0.70	-0.0927193	0.0003103	0.6572807	0.0003103	3.6572807	0.0003103
-0.60	0.0107555	-0.0000746	0.7607555	-0.0000746	3.7607555	-0.0000746
-0.50	0.0947367	-0.0002390	0.8447367	-0.0002390	3.8447367	-0.0002390
-0.40	0.1595462	0.0003377	0.9095462	0.0003377	3.9095462	0.0003377
-0.30	0.2057550	-0.0001784	0.9557550	-0.0001784	3.9557550	-0.0001784
-0.20	0.2344084	-0.0000053	0.9844084	-0.0000053	3.9844084	-0.0000053
-0.10	0.2475960	-0.0000280	0.9975960	-0.0000280	3.9975960	-0.0000280
0.00	0.2500000	0.0051980	1.0000000	0.0051980	4.0000000	0.0051980
0.02	0.2500265	0.0047133	1.0000265	0.0047133	4.0000265	0.0047133
0.04	0.2508321	0.0047295	1.0008321	0.0047295	4.0008321	0.0047295
0.10	0.2545577	-0.0035813	1.0045577	-0.0035813	4.0045577	-0.0035813
0.14	0.2693657	-0.0027534	1.0153657	-0.0027534	4.0153657	-0.0027534
0.18	0.2928703	-0.0004974	1.0428703	-0.0004974	4.0428703	-0.0004974

TABLE 3. - VALUES OF $\tilde{v}_{n,j}$ AND $S(\tilde{v}_{n,j})$.

PSIC 3, 23

TMCETA		0.50		1.00		2.00
M	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	-1.0324975	0.0005133	-1.3149951	0.0010245	3.3700099	0.0020530
-0.90	-0.7882527	0.0004465	-0.8265053	0.0009130	4.3449893	0.0018661
-0.80	-0.5738144	-0.0004957	-0.3976288	-0.0009914	5.2047423	-0.0019829
-0.70	-0.3890790	0.0004655	-0.0281579	0.0009310	5.9436842	0.0018620
-0.60	-0.2338667	-0.0001119	0.2822465	-0.0002239	6.5649331	-0.0004478
-0.50	-0.1078950	-0.0003584	0.5342100	-0.0007169	7.0684201	-0.0014338
-0.40	-0.0104807	0.0005066	0.7286385	0.0010131	7.4572771	0.0020262
-0.30	0.0588324	-0.0002676	0.8672651	-0.0005352	7.7345303	-0.0010703
-0.20	0.1016125	-0.0000079	0.9532251	-0.0000158	7.9064502	-0.0000316
-0.10	0.1213939	-0.0000419	0.9927879	-0.0000039	7.9855758	-0.0001677
0.00	0.1250000	0.0077970	1.0000000	0.0155940	8.0000000	0.0311880
0.02	0.1250398	0.0070700	1.0000796	0.0141399	8.0001592	0.0282799
0.04	0.1252482	0.0070942	1.0024964	0.0141885	8.0049927	0.0283769
0.10	0.1318364	-0.0053720	1.0136731	-0.0107440	8.0273463	-0.0214881
0.14	0.1480485	-0.0041301	1.0460971	-0.0082602	8.0921942	-0.0165205
0.18	0.1893054	-0.0007461	1.1286108	-0.0014923	8.2572217	-0.0029846

PSIC 4, 23

TMCETA		0.50		1.00		2.00
M	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	-0.4745369	-0.0010710	-3.0095295	0.0004688	-1.8994999	0.0066279
-0.90	-0.4626686	-0.0006896	-2.2649266	0.0007299	1.7760413	0.0063282
-0.80	-0.4078118	0.0004281	-1.5667551	-0.0010591	3.0474718	-0.0070076
-0.70	-0.3270985	-0.0001724	-0.9318353	0.0012238	7.8992173	0.0068097
-0.60	-0.2352251	-0.0000048	-0.3743253	-0.0003407	10.3192739	-0.0016840
-0.50	-0.1443976	-0.0001358	0.0944175	-0.0012112	12.2996777	-0.0055125
-0.40	-0.0642527	0.0003233	0.4662051	0.0018430	13.8380364	0.0079216
-0.30	-0.0016996	-0.0002202	0.7366981	-0.0010229	14.9402889	-0.0042339
-0.20	0.0393849	-0.0000074	0.9067245	-0.0000311	15.6260751	-0.0001258
-0.10	0.0589006	-0.0000415	0.9855825	-0.0001673	15.9423098	-0.0006705
0.00	0.0625000	0.0077970	1.0000000	0.0311880	16.0000000	0.1247520
0.02	0.0625398	0.0070707	1.0001592	0.0282806	16.0006368	0.1131203
0.04	0.0637490	0.0071178	1.0049936	0.0284005	16.0199718	0.1135312
0.10	0.0693430	-0.0054700	1.0273727	-0.0215860	16.1094116	-0.0860502
0.14	0.0858594	-0.0043840	1.0925050	-0.0167743	16.3690876	-0.0663357
0.18	0.1293510	-0.0008741	1.2597673	-0.0031126	17.0314324	-0.0120663

PSIC 5, 23

TMCETA		0.50		1.00		2.00
M	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	0.6178204	-0.0035329	-3.6143469	-0.0044996	-23.5285952	0.0115312
-0.90	0.2404163	-0.0024516	-3.1479308	-0.0025706	-12.8259684	0.0135198
-0.80	0.0201411	0.0018943	-2.5162499	0.0013141	-2.9850768	-0.0172004
-0.70	-0.0859480	-0.0012074	-1.8047907	-0.0000874	5.8272606	0.0184448
-0.60	-0.1149516	0.0001745	-1.0867369	-0.0002107	13.4710569	-0.0048991
-0.50	-0.0978357	0.0002578	-0.4226463	-0.0012766	19.8389081	-0.0168910
-0.40	-0.0394971	-0.0000361	0.1401021	0.0024608	24.8529750	0.0251836
-0.30	-0.0186366	-0.0001046	0.5683897	-0.0015471	28.4820821	-0.0137972
-0.20	0.0124463	-0.0000053	0.8454553	-0.0000502	30.7554124	-0.0004161
-0.10	0.0282617	-0.0000339	0.9759931	-0.0002775	31.8077440	-0.0022324
0.00	0.0312500	0.0064975	1.0000000	0.0319800	32.0000000	0.4158401
0.02	0.0312832	0.0058935	1.0002654	0.0471369	32.0021228	0.3770725
0.04	0.0322923	0.0059709	1.0083255	0.0474129	32.0065785	0.3785949
0.10	0.0370131	-0.0047215	1.0457091	-0.0363031	32.3648812	-0.2874870
0.14	0.0512342	-0.0040764	1.1552112	-0.0288034	33.2323641	-0.2228115
0.18	0.0912019	-0.0009418	1.4414309	-0.0056143	35.4550788	-0.0410744

TABLE 3. - VALUES OF $\tilde{\psi}_{n,j}$ AND $S(\tilde{\psi}_{n,j})$.

PSI(6, 2)

THETA	0.50		1.00		2.00	
M	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	1.1133588	-0.0024360	-1.7736232	-0.0154471	-84.4775235	-0.0097500
-0.90	0.6492214	-0.0021585	-2.5622016	-0.0105664	-59.0863189	0.0082856
-0.80	0.3219300	0.0023216	-2.6744262	0.0077666	-34.2139736	-0.0250210
-0.70	0.1151490	-0.0015226	-2.3195610	-0.0043376	-10.8297827	0.0367700
-0.60	0.0046992	0.0002633	-1.6874585	0.0004190	10.2339032	-0.0115520
-0.50	-0.0368860	0.0005176	-0.9434308	-0.0003384	28.2922043	-0.0440877
-0.40	-0.0365965	-0.0003255	-0.2237876	0.0023616	42.8258656	0.0700976
-0.30	-0.0178341	0.0000345	0.3687346	-0.0019712	53.5111731	-0.0399767
-0.20	0.0020316	-0.0000031	0.7702353	-0.0000716	60.2744620	-0.0012337
-0.10	0.0133963	-0.0000247	0.9640398	-0.0004133	63.4234326	-0.0066851
0.00	0.0156250	0.0046731	1.0000000	0.0779700	64.0000000	1.2475203
0.02	0.0156499	0.0044215	1.0003980	0.0707109	64.0043685	1.1312401
0.04	0.0164083	0.0045225	1.0124947	0.0712965	64.1997611	1.1364931
0.06	0.0199970	-0.0037262	1.0687618	-0.0551907	65.0954355	-0.8654006
0.08	0.0312027	-0.0035461	1.2351546	-0.0451218	67.7064243	-0.8760627
0.10	0.0655245	-0.0009453	1.6814011	-0.0094004	74.4417677	-0.1270821

PSI(7, 2)

THETA	0.50		1.00		2.00	
M	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	0.4464706	0.0044380	2.9722533	-0.0259741	-723.9077192	-0.1690738
-0.90	0.4278564	0.0015702	0.0408533	-0.0205603	-175.9867600	-0.0837709
-0.80	0.3034076	-0.0001399	-1.5825563	0.0174627	-122.6816284	0.0207164
-0.70	0.1665806	-0.0005601	-2.1621379	-0.0117616	-66.0965376	0.0379754
-0.60	0.0626002	0.0001802	-1.9955837	0.0017032	-15.8754616	-0.0211136
-0.50	0.0041129	0.0004910	-1.3817143	0.0021209	30.6801310	-0.0995564
-0.40	-0.0156419	-0.0004219	-0.5933567	0.0009936	69.7022726	0.1762525
-0.30	-0.0121201	0.0000698	0.1459684	-0.0021292	98.8728646	-0.1064958
-0.20	-0.0012367	-0.0000014	0.6821422	-0.0000934	117.5991620	-0.0033994
-0.10	0.0062641	-0.0000166	0.9497492	-0.0003730	126.3843603	-0.0186732
0.00	0.0078125	0.0034112	1.0000000	0.1091580	128.0000000	3.4930568
0.02	0.0078299	0.0030964	1.0005573	0.0990058	128.0178320	3.1675563
0.04	0.0083623	0.0032072	1.0175046	0.1001461	128.5594269	3.1844260
0.06	0.0109195	-0.0027837	1.0966369	-0.0786460	131.0701774	-2.4361065
0.08	0.0192794	-0.0029624	1.3336058	-0.0667962	136.4128771	-1.9215505
0.10	0.0476520	-0.0009330	1.9905103	-0.0150590	157.5238794	-0.3703781

PSI(8, 2)

THETA	0.50		1.00		2.00	
M	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	-0.7431922	0.0106611	9.0849024	-0.0170070	-488.1153571	-0.6094140
-0.90	-0.1576505	0.0056536	4.2557871	-0.0223131	-428.3197002	-0.5145511
-0.80	0.0601951	-0.0029789	0.8208215	0.0247473	-335.6181664	0.3165924
-0.70	0.0938495	0.0010005	-1.1527736	-0.0201550	-223.8673185	-0.0941015
-0.60	0.0595074	-0.0000098	-1.8603734	0.0035262	-106.0258446	-0.0213653
-0.50	0.0193905	0.0002468	-1.4511619	0.0063125	4.3884812	-0.1893028
-0.40	-0.0028237	-0.0003460	-0.9328852	-0.0021141	103.6053357	0.4049504
-0.30	-0.0087235	0.0000891	-0.0901534	-0.0018417	179.1288441	-0.2672753
-0.20	-0.0017496	-0.0000003	0.5824978	-0.0001135	228.3644747	-0.0088829
-0.10	0.0028836	-0.0000105	0.9331547	-0.0007546	251.6994570	-0.0496449
0.00	0.0039063	0.0022741	1.0000000	0.1455440	256.0000000	9.3148181
0.02	0.0039179	0.0020454	1.0007431	0.1320253	256.0475523	8.4470969
0.04	0.0042741	0.0021729	1.0233594	0.1340802	257.4921248	8.5016895
0.06	0.0040169	-0.0020053	1.1294681	-0.1071730	264.1970055	-6.5276195
0.08	0.0120356	-0.0024054	1.4521856	-0.0952249	283.6842189	-5.2198640
0.10	0.0349473	-0.0009131	2.3831222	-0.0234227	335.4928553	-1.0368816

TABLE 3. - VALUES OF $\tilde{\psi}_{n,j}$ AND $S(\tilde{\psi}_{n,j})$.

PSI(9, 2)

THETA	0.50		1.00		2.00	
M	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	-1.2133478	0.0054818	12.6652944	0.0365521	-874.4017672	-2.7584680
-0.90	-0.3403117	0.0047903	8.3733405	0.0004562	-894.4433360	-1.9704558
-0.80	-0.1672623	-0.0036097	3.9732432	0.0148278	-788.8941915	1.4595569
-0.70	-0.0099840	0.0018610	0.4386254	-0.0241549	-595.9344597	-0.7607576
-0.60	0.0271185	-0.0001682	-1.2081908	0.0053615	-354.5353439	0.0426525
-0.50	0.0174598	-0.0000354	-1.6800279	0.0118865	-101.8030539	-0.2656524
-0.40	0.0026182	-0.0001902	-1.2082774	-0.0072137	130.5827630	0.8473974
-0.30	-0.0030737	0.0000778	-0.3287490	-0.0009374	316.7469395	-0.6349448
-0.20	-0.0014036	0.0000002	0.4728493	-0.0001293	441.2564767	-0.0222824
-0.10	0.0013031	-0.0000063	0.9142960	-0.0009558	500.8491602	-0.1271948
0.00	0.0019531	0.0014619	1.0000000	0.1871280	512.0000000	23.9523893
0.02	0.0019606	0.0013286	1.0009554	0.1697738	512.1222784	21.7219704
0.04	0.0021905	0.0014238	1.0300644	0.1732418	515.8378791	21.8887130
0.10	0.0033406	-0.0014078	1.1674169	-0.1413877	533.1084816	-16.8986748
0.14	0.0075695	-0.0019110	1.5928936	-0.1321908	584.0629343	-13.7198787
0.18	0.0257881	-0.0008540	2.8778246	-0.0356539	719.9196076	-2.8210583

PSI(10, 2)

THETA	0.50		1.00		2.00	
M	ψ	$S(\psi)$	ψ	$S(\psi)$	ψ	$S(\psi)$
-1.00	-0.4087223	-0.0115214	8.7326581	0.1396283	-1150.9249996	-7.5178146
-0.90	-0.3939215	-0.0022069	9.6536534	0.0595592	-1487.7538305	-5.9946804
-0.80	-0.2108838	-0.0008952	6.8619251	-0.0122068	-1443.9983741	4.9911078
-0.70	-0.0679881	0.0013106	2.8274878	-0.0160981	-1402.5194118	-3.1262375
-0.60	-0.0034079	-0.0001998	-0.0813693	0.0062478	-963.0764736	0.3560739
-0.50	0.0091792	-0.0002085	-1.4258316	0.0177556	-430.0194189	-0.0686977
-0.40	0.0035683	-0.0000429	-1.3806578	-0.0141768	98.9535303	1.5775018
-0.30	-0.0010553	0.0000580	-0.5584002	0.0007237	543.0490865	-1.4379189
-0.20	-0.0009282	0.0000004	0.3549450	-0.0001380	848.0545812	-0.0540891
-0.10	0.0005757	-0.0000036	0.8932192	-0.0011739	996.3953462	-0.3164369
0.00	0.0009766	0.0009137	1.0000000	0.2339101	1024.0000000	59.8809732
0.02	0.0009812	0.0008310	1.0011943	0.2122567	1024.3056990	54.3074487
0.04	0.0011257	0.0009096	1.0376256	0.2177983	1033.5975744	54.8012246
0.10	0.0018665	-0.0009697	1.2106738	-0.1820257	1076.8601577	-42.5781384
0.14	0.0047868	-0.0014913	1.7581425	-0.1799310	1205.2130064	-35.1742668
0.18	0.0191189	-0.0007832	3.4983495	-0.0533627	1552.6287527	-7.5140879

3. Examples of Families of Flow Patterns in the H, θ and Physical Planes.

In this section we construct a number of examples of particular solutions of equation (2.5) in the H, θ plane and we show how to obtain the corresponding flows in the physical plane.

In order to transform a solution from the H, θ plane to the physical plane we make use of the relations

$$x = \int \left[\frac{(M^2-1)\cos \theta}{\rho v^2} \psi_\theta - \frac{\sin \theta}{\rho v} \psi_v \right] dv + \left[\frac{\cos \theta}{\rho} \psi_v - \frac{\sin \theta}{\rho v} \psi_\theta \right] d\theta$$

(3.1)

$$y = \int \left[\frac{(M^2-1)\sin \theta}{\rho v^2} \psi_\theta + \frac{\cos \theta}{\rho v} \psi_v \right] dv + \left[\frac{\sin \theta}{\rho} \psi_v + \frac{\cos \theta}{\rho v} \psi_\theta \right] d\theta$$

We observe that $\psi_v dv + \psi_\theta d\theta = 0$ along a streamline. Hence when computing a streamline $\psi(x,y) = \text{constant}$, if we integrate along the streamline, the transformation equations (3.1) simplify to

$$x = \int \frac{\cos \theta}{\rho} \left[\frac{M^2-1}{v^2} \psi_\theta dv + \psi_v d\theta \right]$$

(3.2)

$$y = \int \frac{\sin \theta}{\rho} \left[\frac{M^2-1}{v^2} \psi_\theta dv + \psi_v d\theta \right] .$$

We note that if the Jacobian of this transformation vanishes at any point the transformation fails to be one-to-one in a neighborhood of such a point and the flow may have no physical significance in the neighborhood. See e.g. [6, p. 17] or [13, p. 311ff.].

The following five particular solutions of the equation (2.5) were investigated:

$$\psi^{(1)} = 5 \psi_{1,1} + \psi_{5,1} + \psi_{4,2}$$

$$\psi^{(2)} = 5 \psi_{1,1} + \psi_{4,1} + \psi_{3,2}$$

$$\psi^{(3)} = 5 \psi_{1,1} + \psi_{4,1} + \psi_{4,2}$$

$$\psi^{(4)} = 5 \psi_{1,1} + \psi_{2,1} + \psi_{3,1} + \psi_{4,1} + \psi_{4,2}$$

$$\psi^{(5)} = 5 \psi_{1,1} + \psi_{3,1} + \psi_{3,2}$$

Table 4 gives sets of points in the H, θ plane at which these solutions assume certain chosen values. The corresponding values of x, y in the physical plane are also given in this table. The streamlines are drawn in both the hodograph or v, θ -plane and in the physical plane in Figures 1-5.

As noted above the transformation from the H, θ -plane to the physical plane may fail to be one-to-one. Consequently the flows represented by the above streamfunctions correspond to physically possible flows only in certain subdomains. This explains why some of the streamlines drawn intersect. It should also be observed that not all of the intersections of the streamlines with the sonic line (shown as a dotted line) correspond to transitions between subsonic and supersonic flow. Those intersections which do correspond to such transitions are indicated on the figures.

$$\tilde{A}(\tau) = \tilde{A}_{AC}^T + \tilde{A}_{S,1} + \tilde{A}_{h,2}$$

[illegible]

$$\bar{v}(2) = 3.37500$$

TABLE A. - VALUES OF COORDINATES ALONG CERTAIN STRAIGHT LINES.

$$\bar{v}(2) = \bar{v}_{1,1} + \bar{v}_{2,1} + \bar{v}_{3,2}$$

M	0	V	X	Y
-0.124021	2.010000	0.759999	-1.443293	29.428884
-0.185160	1.870000	0.750000	0.299104	24.269740
-0.276132	1.770000	0.810510	1.787624	19.022686
-0.402376	1.600000	0.867771	2.418300	16.113900
-0.510000	1.350000	0.907194	2.381348	13.134431
0.000000	1.000000	0.912971	2.268344	12.561737
0.090000	0.850000	0.901099	2.084191	11.592723
0.160775	0.750000	0.886618	2.111757	10.602342
0.243784	0.600000	0.869516	2.053907	9.653957
0.378098	0.400000	0.848419	1.812403	8.341831
0.500000	0.200000	0.822750	1.725868	6.119927
0.6120376	0.100000	0.791766	1.700258	4.234551
0.717043	0.000000	0.750000	1.700000	2.596623
0.819711	0.000000	0.700000	1.700000	1.700000
0.910000	0.000000	0.650000	1.612344	10.000000

$$\bar{v}(2) = 1.00000$$

$$\bar{v}(2) = 0.12500$$

M	0	V	X	Y
-0.108264	2.010000	0.499492	2.324731	31.308271
-0.178327	1.850000	0.708853	4.510724	25.801011
-0.265013	1.770000	0.740448	5.493156	20.883350
-0.380895	1.650000	0.740924	6.460437	16.785773
-0.510000	1.500000	0.733127	6.553236	13.567020
-0.610967	1.400000	0.716185	6.276874	10.369965
-0.680493	1.250000	0.685354	5.761480	8.100360
-0.752034	1.000000	0.642727	5.134936	6.319600
-0.816363	0.850000	0.608995	4.479814	4.979240
0.000000	0.700000	0.577903	4.203723	4.533949
0.090000	0.550000	0.546619	3.843656	4.147762
0.178327	0.400000	0.516023	3.446623	3.750963
0.265013	0.250000	0.486024	3.106054	3.430284
0.380895	0.100000	0.456512	2.862116	3.165602
0.510000	0.000000	0.427311	2.616131	2.94274
0.610967	0.000000	0.398328	2.36918	2.661950
0.680493	0.000000	0.369345	2.123379	2.387728
0.752034	0.000000	0.340362	1.87853	2.23222
0.816363	0.000000	0.311371	1.63367	2.14114
0.910000	0.000000	0.282388	1.38883	2.05762
1.000000	0.000000	0.253405	1.14400	1.99091
1.090000	0.000000	0.224422	0.89916	1.93772
1.180000	0.000000	0.195439	0.65437	1.89255
1.270000	0.000000	0.166456	0.40958	1.85795
1.360000	0.000000	0.137473	0.16479	1.83130
1.450000	0.000000	0.108490	0.070254	
1.540000	0.000000	0.079507		
1.630000	0.000000	0.050524		
1.720000	0.000000	0.021541		
1.810000	0.000000	0.000000		
1.900000	0.000000	0.000000		
2.000000	0.000000	0.000000		
2.100000	0.000000	0.000000		
2.200000	0.000000	0.000000		
2.300000	0.000000	0.000000		
2.400000	0.000000	0.000000		
2.500000	0.000000	0.000000		
2.600000	0.000000	0.000000		
2.700000	0.000000	0.000000		
2.800000	0.000000	0.000000		
2.900000	0.000000	0.000000		
3.000000	0.000000	0.000000		
3.100000	0.000000	0.000000		
3.200000	0.000000	0.000000		
3.300000	0.000000	0.000000		
3.400000	0.000000	0.000000		
3.500000	0.000000	0.000000		
3.600000	0.000000	0.000000		
3.700000	0.000000	0.000000		
3.800000	0.000000	0.000000		
3.900000	0.000000	0.000000		
4.000000	0.000000	0.000000		
4.100000	0.000000	0.000000		
4.200000	0.000000	0.000000		
4.300000	0.000000	0.000000		
4.400000	0.000000	0.000000		
4.500000	0.000000	0.000000		
4.600000	0.000000	0.000000		
4.700000	0.000000	0.000000		
4.800000	0.000000	0.000000		
4.900000	0.000000	0.000000		
5.000000	0.000000	0.000000		
5.100000	0.000000	0.000000		
5.200000	0.000000	0.000000		
5.300000	0.000000	0.000000		
5.400000	0.000000	0.000000		
5.500000	0.000000	0.000000		
5.600000	0.000000	0.000000		
5.700000	0.000000	0.000000		
5.800000	0.000000	0.000000		
5.900000	0.000000	0.000000		
6.000000	0.000000	0.000000		
6.100000	0.000000	0.000000		
6.200000	0.000000	0.000000		
6.300000	0.000000	0.000000		
6.400000	0.000000	0.000000		
6.500000	0.000000	0.000000		
6.600000	0.000000	0.000000		
6.700000	0.000000	0.000000		
6.800000	0.000000	0.000000		
6.900000	0.000000	0.000000		
7.000000	0.000000	0.000000		
7.100000	0.000000	0.000000		
7.200000	0.000000	0.000000		
7.300000	0.000000	0.000000		
7.400000	0.000000	0.000000		
7.500000	0.000000	0.000000		
7.600000	0.000000	0.000000		
7.700000	0.000000	0.000000		
7.800000	0.000000	0.000000		
7.900000	0.000000	0.000000		
8.000000	0.000000	0.000000		
8.100000	0.000000	0.000000		
8.200000	0.000000	0.000000		
8.300000	0.000000	0.000000		
8.400000	0.000000	0.000000		
8.500000	0.000000	0.000000		
8.600000	0.000000	0.000000		
8.700000	0.000000	0.000000		
8.800000	0.000000	0.000000		
8.900000	0.000000	0.000000		
9.000000	0.000000	0.000000		
9.100000	0.000000	0.000000		
9.200000	0.000000	0.000000		
9.300000	0.000000	0.000000		
9.400000	0.000000	0.000000		
9.500000	0.000000	0.000000		
9.600000	0.000000	0.000000		
9.700000	0.000000	0.000000		
9.800000	0.000000	0.000000		
9.900000	0.000000	0.000000		
10.000000	0.000000	0.000000		

TABLE 4. - VALUES OF COORDINATES ALONG CERTAIN STRENGTHS.

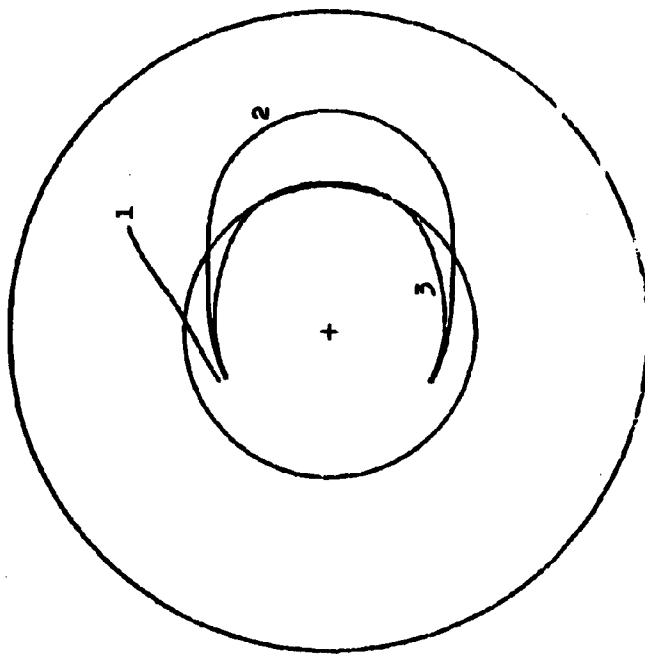
$$\bar{r}^{(k)} = \bar{r}_{1,1}^{(k)} + \bar{r}_{2,1}^{(k)} + \bar{r}_{3,1}^{(k)} + \bar{r}_{4,1}^{(k)} + \bar{r}_{5,1}^{(k)}$$

M	N	O	X	Y
-0.872254	-0.203386	2.010000	0.466274	0.471703
-0.212254	-0.101700	1.890000	0.724392	-0.766404
-0.212254	-0.110854	1.770000	0.760448	1.495799
-0.212254	-0.067661	1.650000	0.830893	2.103103
-0.212254	-0.014106	1.530000	0.892607	2.294079
-0.187287	0.000000	1.410000	0.912071	2.430109
-0.187287	0.024282	1.290000	0.900070	2.460093
-0.082490	0.054367	1.170000	0.900019	2.460093
-0.082490	0.000000	1.050000	1.000021	1.553000
-0.115400	0.115400	0.930000	1.121801	1.553000
0.115400	0.115400	0.810000	1.243601	1.553000
0.170000	0.170000	0.690000	1.365401	1.553000
0.224600	0.224600	0.570000	1.487201	1.553000
0.279200	0.279200	0.450000	1.609001	1.553000
0.333800	0.333800	0.330000	1.730801	1.553000
0.388400	0.388400	0.210000	1.852601	1.553000
0.443000	0.443000	0.090000	1.974401	1.553000
0.497600	0.497600	-0.030000	2.096201	1.553000
0.552200	0.552200	-0.150000	2.218001	1.553000
0.606800	0.606800	-0.270000	2.339801	1.553000
0.661400	0.661400	-0.390000	2.461601	1.553000
0.716000	0.716000	-0.510000	2.583401	1.553000
0.770600	0.770600	-0.630000	2.705201	1.553000
0.825200	0.825200	-0.750000	2.827001	1.553000
0.879800	0.879800	-0.870000	2.948801	1.553000
0.934400	0.934400	-0.990000	3.070601	1.553000
0.989000	0.989000	-1.110000	3.192401	1.553000
1.043600	1.043600	-1.230000	3.314201	1.553000
1.098200	1.098200	-1.350000	3.436001	1.553000
1.152800	1.152800	-1.470000	3.557801	1.553000
1.207400	1.207400	-1.590000	3.679601	1.553000
1.262000	1.262000	-1.710000	3.801401	1.553000
1.316600	1.316600	-1.830000	3.923201	1.553000
1.371200	1.371200	-1.950000	4.045001	1.553000
1.425800	1.425800	-2.070000	4.166801	1.553000
1.480400	1.480400	-2.190000	4.288601	1.553000
1.535000	1.535000	-2.310000	4.410401	1.553000
1.589600	1.589600	-2.430000	4.532201	1.553000
1.644200	1.644200	-2.550000	4.654001	1.553000
1.698800	1.698800	-2.670000	4.775801	1.553000
1.753400	1.753400	-2.790000	4.897601	1.553000
1.808000	1.808000	-2.910000	5.019401	1.553000
1.862600	1.862600	-3.030000	5.141201	1.553000
1.917200	1.917200	-3.150000	5.263001	1.553000
1.971800	1.971800	-3.270000	5.384801	1.553000
2.026400	2.026400	-3.390000	5.506601	1.553000
2.081000	2.081000	-3.510000	5.628401	1.553000
2.135600	2.135600	-3.630000	5.750201	1.553000
2.190200	2.190200	-3.750000	5.872001	1.553000
2.244800	2.244800	-3.870000	5.993801	1.553000
2.299400	2.299400	-3.990000	6.115601	1.553000
2.354000	2.354000	-4.110000	6.237401	1.553000
2.408600	2.408600	-4.230000	6.359201	1.553000
2.463200	2.463200	-4.350000	6.481001	1.553000
2.517800	2.517800	-4.470000	6.602801	1.553000
2.572400	2.572400	-4.590000	6.724601	1.553000
2.627000	2.627000	-4.710000	6.846401	1.553000
2.681600	2.681600	-4.830000	6.968201	1.553000
2.736200	2.736200	-4.950000	7.090001	1.553000
2.790800	2.790800	-5.070000	7.211801	1.553000
2.845400	2.845400	-5.190000	7.333601	1.553000
2.899999	2.899999	-5.310000	7.455401	1.553000
2.954599	2.954599	-5.430000	7.577201	1.553000
3.009199	3.009199	-5.550000	7.699001	1.553000
3.063799	3.063799	-5.670000	7.820801	1.553000
3.118399	3.118399	-5.790000	7.942601	1.553000
3.172999	3.172999	-5.910000	8.064401	1.553000
3.227599	3.227599	-6.030000	8.186201	1.553000
3.282199	3.282199	-6.150000	8.308001	1.553000
3.336799	3.336799	-6.270000	8.429801	1.553000
3.391399	3.391399	-6.390000	8.551601	1.553000
3.445999	3.445999	-6.510000	8.673401	1.553000
3.500599	3.500599	-6.630000	8.795201	1.553000
3.555199	3.555199	-6.750000	8.917001	1.553000
3.609799	3.609799	-6.870000	9.038801	1.553000
3.664399	3.664399	-6.990000	9.160601	1.553000
3.718999	3.718999	-7.110000	9.282401	1.553000
3.773599	3.773599	-7.230000	9.404201	1.553000
3.828199	3.828199	-7.350000	9.526001	1.553000
3.882799	3.882799	-7.470000	9.647801	1.553000
3.937399	3.937399	-7.590000	9.769601	1.553000
3.991999	3.991999	-7.710000	9.891401	1.553000
4.046599	4.046599	-7.830000	10.013201	1.553000
4.101199	4.101199	-7.950000	10.135001	1.553000
4.155799	4.155799	-8.070000	10.256801	1.553000
4.210399	4.210399	-8.190000	10.378601	1.553000
4.264999	4.264999	-8.310000	10.500401	1.553000
4.319599	4.319599	-8.430000	10.622201	1.553000
4.374199	4.374199	-8.550000	10.744001	1.553000
4.428799	4.428799	-8.670000	10.865801	1.553000
4.483399	4.483399	-8.790000	10.987601	1.553000
4.537999	4.537999	-8.910000	11.109401	1.553000
4.592599	4.592599	-9.030000	11.231201	1.553000
4.647199	4.647199	-9.150000	11.353001	1.553000
4.701799	4.701799	-9.270000	11.474801	1.553000
4.756399	4.756399	-9.390000	11.596601	1.553000
4.810999	4.810999	-9.510000	11.718401	1.553000
4.865599	4.865599	-9.630000	11.840201	1.553000
4.920199	4.920199	-9.750000	11.962001	1.553000
4.974799	4.974799	-9.870000	12.083801	1.553000
5.029399	5.029399	-9.990000	12.205601	1.553000
5.083999	5.083999	-10.110000	12.327401	1.553000
5.138599	5.138599	-10.230000	12.449201	1.553000
5.193199	5.193199	-10.350000	12.571001	1.553000
5.247799	5.247799	-10.470000	12.692801	1.553000
5.302399	5.302399	-10.590000	12.814601	1.553000
5.356999	5.356999	-10.710000	12.936401	1.553000
5.411599	5.411599	-10.830000	13.058201	1.553000
5.466199	5.466199	-10.950000	13.180001	1.553000
5.520799	5.520799	-11.070000	13.301801	1.553000
5.575399	5.575399	-11.190000	13.423601	1.553000
5.629999	5.629999	-11.310000	13.545401	1.553000
5.684599	5.684599	-11.430000	13.667201	1.553000
5.739199	5.739199	-11.550000	13.789001	1.553000
5.793799	5.793799	-11.670000	13.910801	1.553000
5.848399	5.848399	-11.790000	14.032601	1.553000
5.902999	5.902999	-11.910000	14.154401	1.553000
5.957599	5.957599	-12.030000	14.276201	1.553000
6.012199	6.012199	-12.150000	14.398001	1.553000
6.066799	6.066799	-12.270000	14.519801	1.553000
6.121399	6.121399	-12.390000	14.641601	1.553000
6.175999	6.175999	-12.510000	14.763401	1.553000
6.230599	6.230599	-12.630000	14.885201	1.553000
6.285199	6.285199	-12.750000	15.007001	1.553000
6.339799	6.339799	-12.870000	15.128801	1.553000
6.394399	6.394399	-12.990000	15.250601	1.553000
6.448999	6.448999	-13.110000	15.372401	1.553000
6.503599	6.503599	-13.230000	15.494201	1.553000
6.558199	6.558199	-13.350000	15.616001	1.553000
6.612799	6.612799	-13.470000	15.737801	1.553000
6.667399	6.667399	-13.590000	15.859601	1.553000
6.721999	6.721999	-13.710000	15.981401	1.553000
6.776599	6.776599	-13.830000	16.103201	1.553000
6.831199	6.831199	-13.950000	16.225001	1.553000
6.885799	6.885799	-14.070000	16.346801	1.553000
6.940399	6.940399	-14.190000	16.468601	1.553000
6.994999	6.994999	-14.310000	16.590401	1.553000
7.049599	7.049599	-14.430000	16.712201	1.553000
7.104199	7.104199	-14.550000	16.834001	1.553000
7.158799	7.158799	-14.670000	16.955801	1.553000
7.213399	7.213399	-14.790000	17.077601	1.553000
7.267999	7.267999	-14.910000	17.199401	1.553000
7.322599	7.322599	-15.030000	17.321201	1.553000
7.377199	7.377199	-15.150000	17.443001	1.553000
7.431799	7.431799	-15.270000	17.564801	1.553000
7.486399	7.486399	-15.390000	17.686601	1.553000
7.540999	7.540999	-15.510000	17.808401	1.553000
7.595599	7.595599	-15.630000	17.930201	1.553000
7.650199	7.650199	-15.750000	18.052001	1.553000
7.704799	7.704799	-15.870000	18.173801	1.553000
7.759399	7.759399	-15.990000	18.295601	1.553000
7.813999	7.813999	-16.110000	18.417401	1.553000
7.868599	7.868599	-16.230000	18.539201	1.553000
7.923199	7.923199	-16.350000	18.661001	1.553000
7.977799	7.977799	-16.470000	18.782801	1.553000
8.032399	8.032399	-16.590000	18.904601	1.553000
8.086999	8.086999	-16.710000	19.026401	1.553000
8.141599	8.141599	-16.830000	19.148201	1.553000
8.196199	8.196199	-16.950000	19.270001	1.553000
8.250799	8.250799	-17.070000	19.391801	1.553000
8.305399	8.305399	-17.190000	19.513601	1.553000
8.359999	8.359999	-17.310000	19.635401	1.553000
8.414599	8.414599	-17.430000	19.757201	1.553000
8.469199	8.469199	-17.550000	19.879001	1.553000
8.523799	8.523799	-17.670000	19.999801	1.553000
8.578399	8.578399	-17.790000	20.121601	1.553000
8.632999	8.632999	-17.910000	20.243401	1.553000
8.687599	8.687599	-18.030000	20.365201	1.553000
8.742199	8.742199	-18.150000	20.487001	1.553000
8.796799	8.796799	-18.270000	20.608801	1.553000
8.851399	8.851399	-18.390000	20.730601	1.553000
8.905999	8.905999			

TABLE 4. - VALUES OF COORDINATES ALONG CERTAIN STATIONS.

$$\tilde{v}(5) = \tilde{v}_{1,1} + \tilde{v}_{3,1} + \tilde{v}_{3,2}$$

[illegible][illegible]



HODOGRAPH PLANE

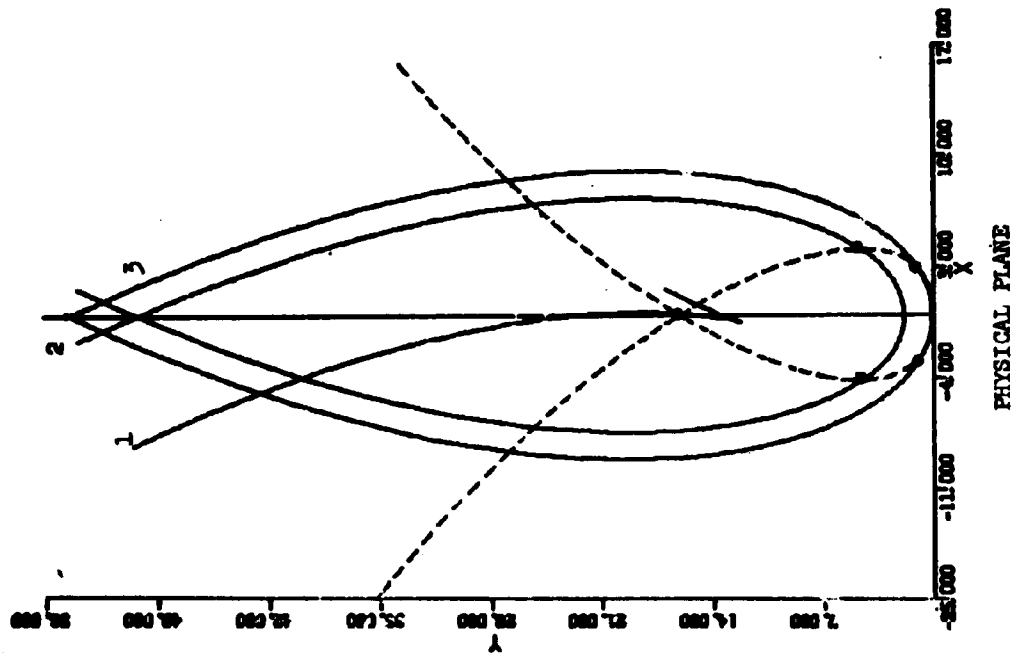
$$\text{Fig. 1 - } \tilde{V}(1) = 5\tilde{V}_{1,1} + \tilde{V}_{5,1} + \tilde{V}_{4,2}$$

$$1 \quad \tilde{V}(1) = 5.0625$$

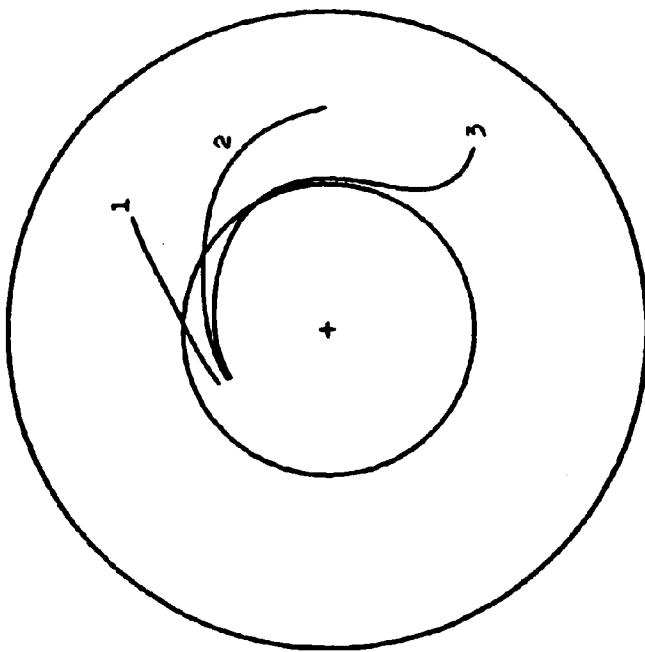
$$2 \quad \tilde{V}(1) = 1.0000$$

$$3 \quad \tilde{V}(1) = 0.0625$$

----- Sonic Line
 • Transition Points



PHYSICAL PLANE



HODOGRAPH PLANE

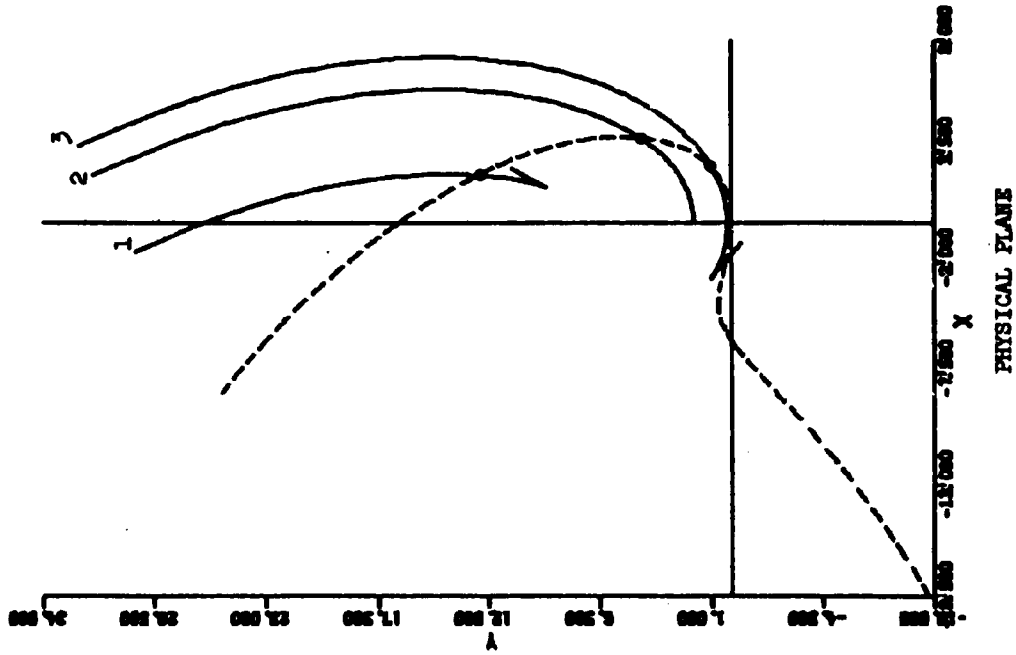
$$\text{Fig. 2 - } \tilde{v}(2) = 5\tilde{v}_{1,1} + \tilde{v}_{4,1} + \tilde{v}_{3,2}$$

$$1 \quad \tilde{v}(2) = 3.375$$

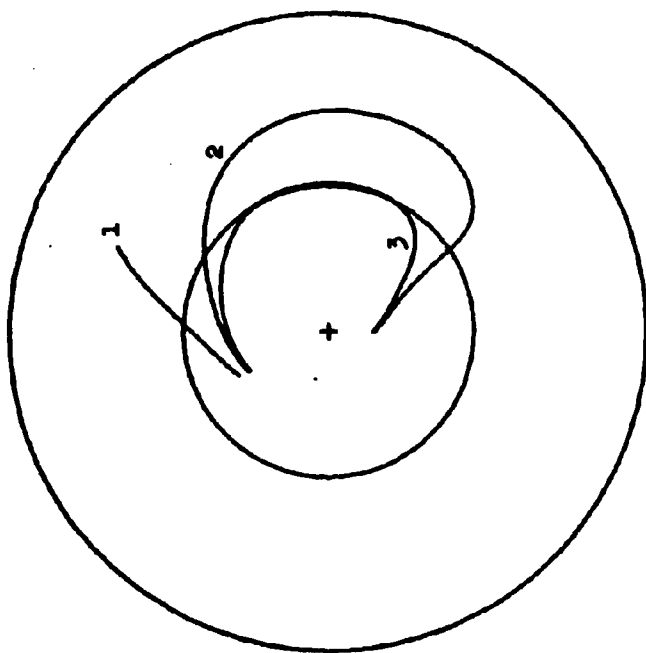
$$2 \quad \tilde{v}(2) = 1.000$$

$$3 \quad \tilde{v}(2) = 0.125$$

----- Sonic Line
 • Transition Points



PHYSICAL PLANE



HODOGRAPH PLANE

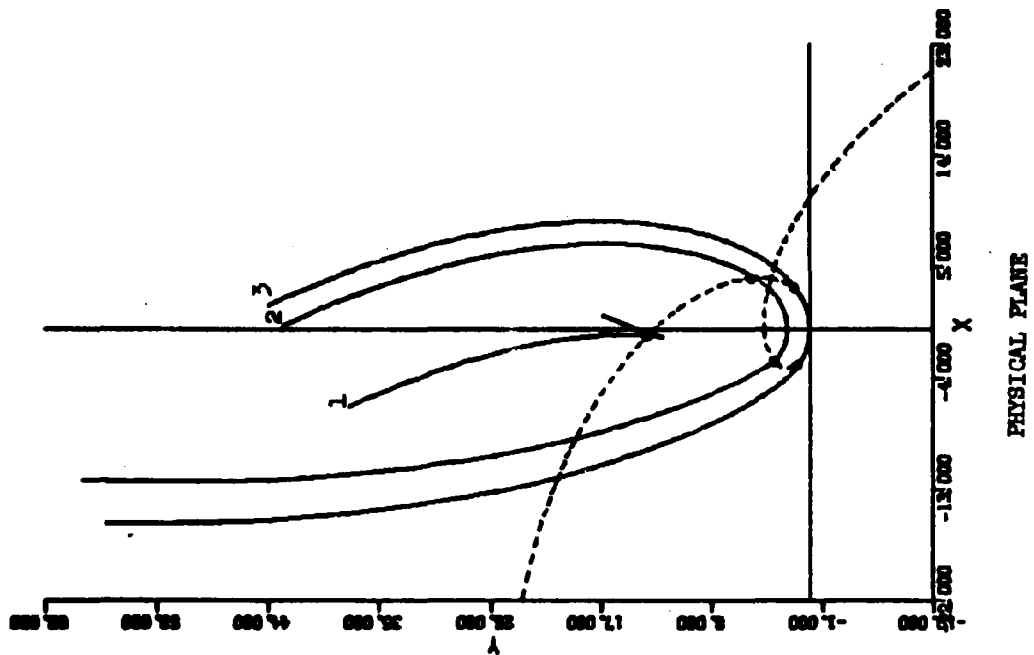
$$\text{Fig. 3 - } \tilde{v}^{(3)} = 5\tilde{v}_{1,1} + \tilde{v}_{4,1} + \tilde{v}_{4,2}$$

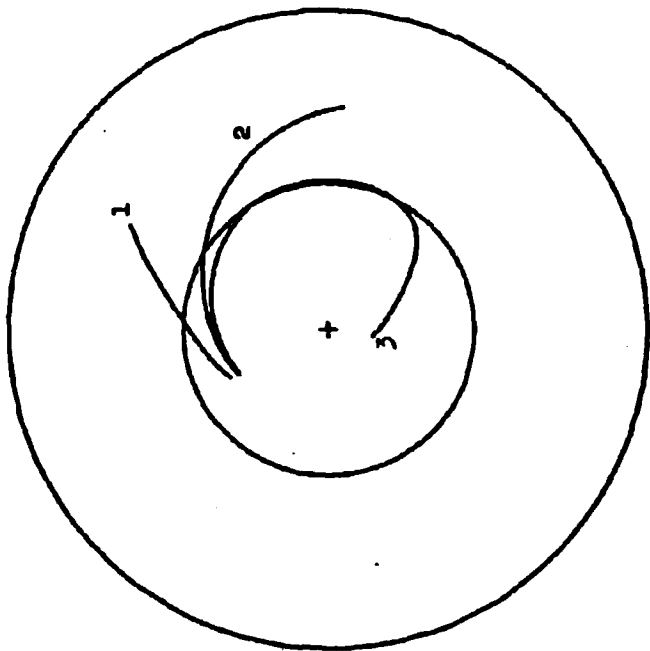
$$1 \quad \tilde{v}^{(3)} = 5.0625$$

$$2 \quad \tilde{v}^{(3)} = 1.0000$$

$$3 \quad \tilde{v}^{(3)} = 0.0625$$

----- Sonic Line
 • Transition Points





HODOGRAPH PLANE

$$\text{Fig. 4} \quad \tilde{v}(k) = \tilde{v}_{1,1} + \tilde{v}_{2,1} + \tilde{v}_{3,1} + \tilde{v}_{4,1} + \tilde{v}_{4,2}$$

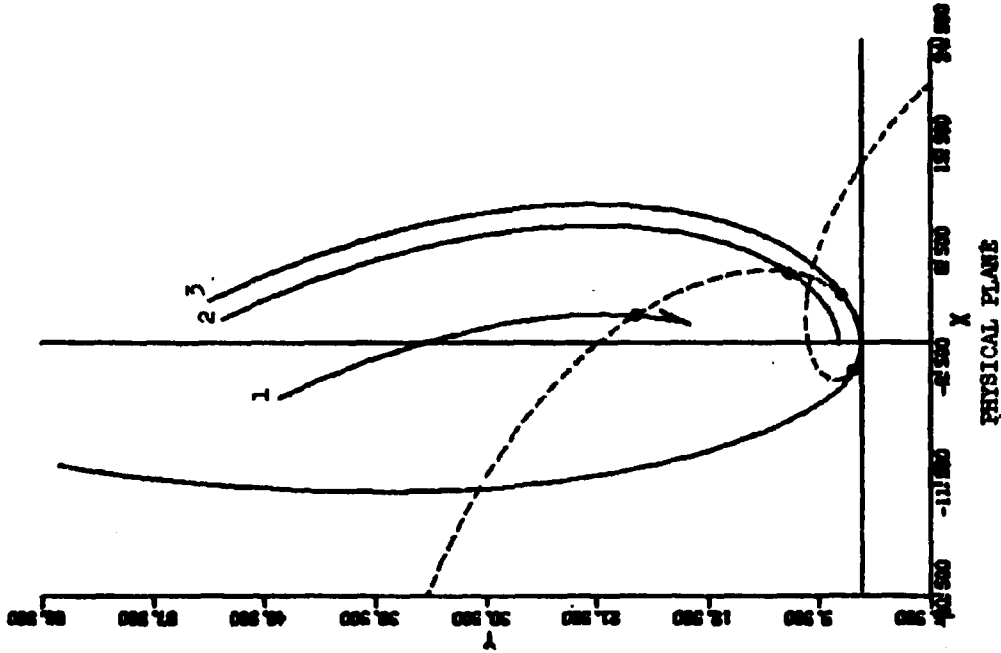
$$1. \quad \tilde{v}(k) = 5.0625$$

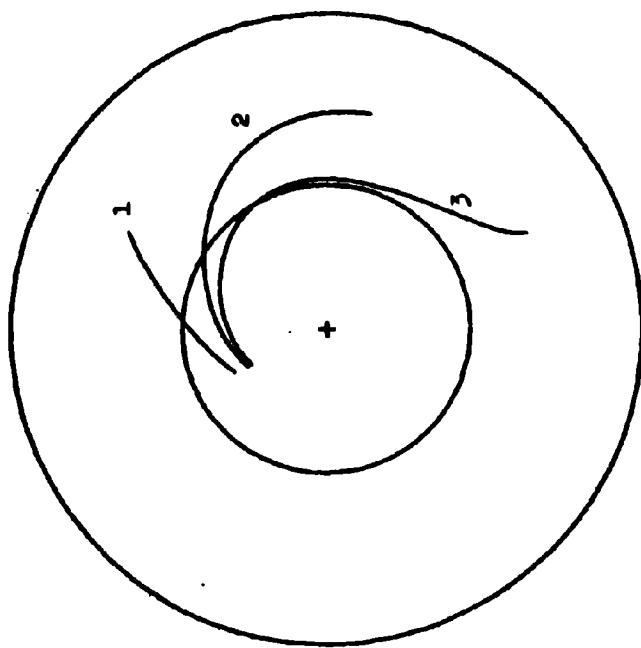
$$2. \quad \tilde{v}(k) = 1.0000$$

$$3. \quad \tilde{v}(k) = 0.0625$$

----- Sonic Line

o Transition Points



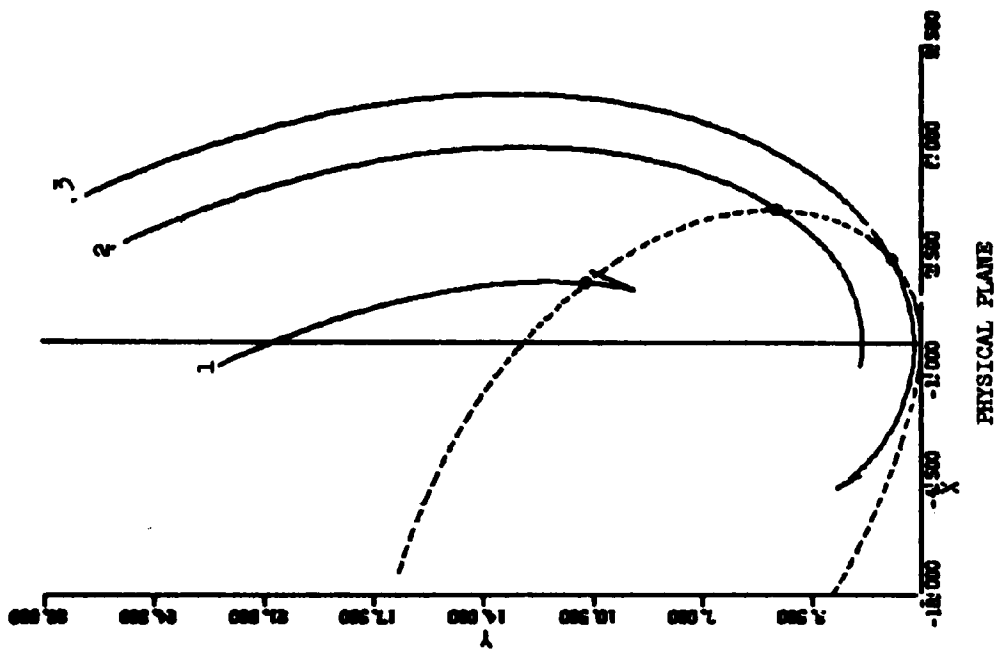


HODOGRAPH PLANE

$$\text{Fig. 5} - \tilde{\varphi}(5) = 5\tilde{\varphi}_{1,1} + \tilde{\varphi}_{3,1} + \tilde{\varphi}_{3,2}$$

- 1 $\tilde{\varphi}(5) = 3.375$
- 2 $\tilde{\varphi}(5) = 1.000$
- 3 $\tilde{\varphi}(5) = 0.125$

----- Sonic Line
 • Transition Points



REFERENCES

1. Bergman, S. The hodograph method in the theory of compressible fluids. Supplement to "Fluid Dynamics" by R. von Mises and K. O. Friedrichs, Brown University, Providence, R. I., 1942.
2. Bergman, S. A formula for the stream function of certain flows, Proc. Nat. Acad. Sci. U.S.A. 29, 276-281 (1943).
3. Bergman, S. Two-dimensional transonic flow patterns, Amer. J. Math. 70, 856-891 (1948).
4. Bergman, S. On representation of stream functions of subsonic and supersonic flows of compressible fluids. J. Ratl. Mech. Anal. 4, 883-905 (1955).
5. Bergman, S. Integral operators in the theory of linear partial differential equations. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, vol. 23. Springer, Berlin-Göttingen-Heidelberg, 1961.
6. Bers, L. Mathematical aspects of subsonic and transonic gas dynamics, Wiley, New York, 1958.
7. Bers, L. and Gelbart, A. On a class of differential equations in the mechanics of continua, Quarterly of Applied Mathematics, 1, 168-188 (1943).
8. Bers, L. and Gelbart, A. On a class of functions defined by partial differential equations, Trans. Amer. Math. Soc. 56, 67-93 (1944).
9. Chaplygin, S. A. On gas jets, Scientific Memoirs, Moscow University

Mathematics-Physics Section, 21, 1-127 (1904) (Eng. Trans.
Pub. by NACA TM 1063, 1944).

10. Golub, G. H. and Smith, L. B. Chebyshev approximation of continuous functions by a Chebyshev's system of functions, (to appear).
11. Legendre, R. Fonctions algébriques et fonctions abéliennes, Office National d'Études et de Recherches Aéronautiques, No. 100, France (1961).
12. Legendre, R. Fonctions abéliennes d'un groupe Kleinéen. Applications à l'aérodynamique, Office National d'Études et de Recherches Aéronautiques, No. 109, France (1962).
13. von Mises, R. Mathematical theory of compressible fluid flow. Academic Press, New York 1958.
14. Molenbroek, P. Über einige Bewegungen eines Gases mit Annahme eines Geschwindigkeitspotentials, Archiv der Math. und Physik (2), 9, 157-195 (1890).
15. Nazarov, G. I. On the exact solutions of some problems of magnetohydrodynamics, Prikl. Mat. Meh. 2, 63-73 (1963) (in Russian).
16. Nazarov, G. I. Solution of a vortex problem in a magnetic field parallel to the velocity of motion, for compressible fluids. Trudy Tomsk. Gos. Univ. Ser. Meh. - Mat. Geom. Sb. 163, 106-114 (1963) (in Russian).
17. Remez, E. Y. General computational methods of Chebyshev approximation. The problems with linear real parameters. Izdatel'stvo Akademiya

Nauk Ukrainskoi^v, S.S.R. Kiev, 1957. AEC-tr-4491. Books 1
and 2, English Translation by U.S. AEC.

18. Stark, J. M. Application of Bergman's integral operators to
transonic flows. Int. J. Non-linear Mech. 1, 17-34, (1966)
19. Stark, J. M. Transonic flow patterns generated by Bergman's integral
operator. Report, Department of Mathematics, Stanford University
(1964), 1 - 92.